

Nonlinear Phenomena in Canonical Stochastic Quantization

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Abstract

Stochastic quantization provides a connection between quantum field theory and statistical mechanics, with applications especially in gauge field theories. Euclidean quantum field theory is viewed as the equilibrium limit of a statistical system coupled to a thermal reservoir. Nonlinear phenomena in stochastic quantization arise when employing nonlinear Brownian motion as an underlying stochastic process. We discuss a novel formulation of the Higgs mechanism in QED.

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1 Introduction

Great interest lies in a nonperturbatively valid quantization procedure for Yang–Mills fields. Let $f = f(A)$ be a gauge invariant observable, then

$$\langle f \rangle \stackrel{?}{=} \frac{\overbrace{\int DA e^{-S_{inv}[A]} f(A)}^{\infty}}{\underbrace{\int DA e^{-S_{inv}[A]}}_{\infty}} \quad (1.1)$$

or rather

$$\langle f \rangle \stackrel{?}{=} \frac{\int DB |det d^* D_B| e^{-S_{inv}[B]} f(B) \overbrace{\int Dg}^{\infty}}{\int DB |det d^* D_B| e^{-S_{inv}[B]} \underbrace{\int Dg}_{\infty}} \quad (1.2)$$

is not well defined and requires a definite meaning. Here S_{inv} denotes the gauge invariant Yang–Mills action, B fulfills the gauge condition $\partial B = 0$, $det d^* D_B$ denotes the Faddeev–Popov determinant. Formally cancelling the infinite gauge group volume we obtain the Faddeev–Popov formula

$$\langle f \rangle \stackrel{?}{=} \frac{\int DB |det d^* D_B| e^{-S[B]} f(B)}{\int DB |det d^* D_B| e^{-S[B]}}. \quad (1.3)$$

From [Gribov, 1978] we know, however, that gauge fixing is not unique and that $det d^* D_B$ may vanish. Two related issues have to be addressed: the infinite gauge group volume and non-uniqueness of the gauge fixing procedure. We do not attempt to review here the many investigations of the Gribov problem. Among the various proposals for improving the Yang–Mills path integral we just mention the stochastic quantization scheme:

- the stochastic quantization scheme is intrinsically well defined [Parisi & Wu, 1981]
- a globally valid path integral is possible [Hüffel & Kelnhofer, 2000]

2 Stochastic Quantization

Consider Euclidean scalar field theory with action $S[\Phi]$. The main idea of stochastic quantization is to view Euclidean quantum field theory as the equilibrium limit of a statistical system coupled to a thermal reservoir. This system evolves in a new additional time direction s which is called stochastic time until it reaches the equilibrium limit for infinite stochastic time. In the equilibrium limit the stochastic averages become identical to ordinary Euclidean vacuum expectation values.

We remark that the Parisi-Wu stochastic quantization scheme requires the introduction of an additional time variable s as opposed to the methods of Nelson's stochastic mechanics [Nelson, 1967] applied to the field theory case [Guerra & Ruggiero, 1973].

There are two equivalent formulations of stochastic quantization: In one formulation the stochastic process is introduced in such a way that its probability density ρ converges for $s \rightarrow \infty$ towards the path integral density

$$\rho[\Phi, s] \longrightarrow \frac{e^{-S[\Phi]}}{\int D\Phi e^{-S[\Phi]}}. \quad (2.1)$$

This scenario is implemented by introducing the Smoluchovski equation

$$\frac{\partial \rho[\Phi, s]}{\partial s} = \int d^4x \frac{\delta}{\delta \Phi(x)} \left[\frac{\delta S}{\delta \Phi(x)} + \frac{\delta}{\delta \Phi(x)} \right] \rho[\Phi, s], \quad (2.2)$$

Green functions are obtained in the equilibrium limit as

$$\langle f \rangle = \lim_{s \rightarrow \infty} \int D\Phi f(\Phi) \rho[\Phi, s]. \quad (2.3)$$

In the second formulation all fields have an additional dependence on the stochastic time $\Phi = \Phi(x, s)$. Their stochastic time evolution is determined by the Langevin equation

$$d\Phi = -\frac{\delta S}{\delta \Phi} ds + dW \quad (2.4)$$

and expectation values of observables are obtained by ensemble averages over the increments of a Wiener process

$$\langle \langle dW(x, s) dW(x', s) \rangle \rangle = 2\delta^4(x - x') ds. \quad (2.5)$$

Similarly as above the Green functions are obtained from

$$\langle f \rangle = \lim_{s \rightarrow \infty} \langle \langle f(\Phi(\cdot, s)) \rangle \rangle. \quad (2.6)$$

3 Stochastic Quantization of Gauge Theories

One of the most interesting aspects of the stochastic quantization scheme lies in its rather unconventional treatment of gauge field theories, in specific of Yang-Mills theories. Originally it was formulated by [Parisi & Wu, 1981] with a Langevin equation

$$dA = -\frac{\delta S_{inv}}{\delta A} ds + dW \quad (3.1)$$

in terms of the gauge invariant Yang-Mills action S_{inv} . There are no gauge fixing terms and no ghost fields introduced. Whereas gauge invariant observables equilibrate, gauge *non*-invariant observables diverge for $s \rightarrow \infty$. This is understood as a consequence of the drift force $-\frac{\delta S}{\delta A}$ acting orthogonal to the gauge orbits. Due to diffusion along the gauge orbits the solution ρ of the associated Smoluchovski equation is not normalizable; no immediate probabilistic interpretation and no immediate path integral formulation are existing.

One would like to introduce an additional conservative damping force along the gauge orbits, which, however, is known to be impossible. It was proposed [Hüffel and Kelnhofer, 1998] to study an equivalence class of stochastic processes by modifying both the drift and the diffusion term, yet leaving gauge invariant variables unchanged. Selecting a geometrically distinguished representative an equilibrium distribution can be derived by inspection.

Adapted coordinates $\Psi = (B, g)$ enable to separate gauge independent and gauge dependent degrees of freedom. They are introduced by $A = B^g$, where B lies in the gauge fixing surface Γ

$$\Gamma = \{ B \mid d^* B = 0 \}, \quad (3.2)$$

$g \in \mathcal{G}$ and B^g is the gauge transformed field

$$B^g = g^{-1} B g + g^{-1} dg. \quad (3.3)$$

The Parisi–Wu Langevin equation in adapted coordinates Ψ obtains as

$$d\Psi = \left(-G^{-1} \frac{\delta S}{\delta \Psi} + \frac{1}{\sqrt{\det G}} \frac{\delta(G^{-1} \sqrt{\det G})}{\delta \Psi} \right) ds + E dW \quad (3.4)$$

where

$$E = \frac{\partial(B, g)}{\partial A}, \quad G^{-1} = EE^*, \quad \sqrt{\det G} \propto \det d^* D_B. \quad (3.5)$$

We consider the equivalence class of Langevin equations

$$d\Psi = \left(-G^{-1} \frac{\delta S}{\delta \Psi} + \frac{1}{\sqrt{\det G}} \frac{\delta(G^{-1} \sqrt{\det G})}{\delta \Psi} + ED_A \mathbf{X} \right) ds + E(\mathbf{1} + D_A \mathbf{Y}) dW. \quad (3.6)$$

Here X and Y contribute only to drift and diffusion terms of the unphysical g -field Langevin equation. X and Y can be shown to be absent in the B -field Langevin equation so that gauge invariant observables remain unaffected. The induced vielbein and the corresponding metric tensor are

$$\tilde{E} = E(\mathbf{1} + D_A \mathbf{Y}), \quad \tilde{G}^{-1} = \tilde{E} \tilde{E}^* \quad (3.7)$$

and we denote by \tilde{g} the pullback of the metric tensor \tilde{G} to the original variables.

We determine Y by demanding that - with respect to \tilde{g} - the gauge orbit becomes orthogonal to the gauge fixing surface! X is obtained - modulo a judiciously chosen Ito term - as a gradient of an arbitrary function $S_{\mathcal{G}}[g]$, fulfilling

$$\int_{\mathcal{G}} \mathcal{D}g \, e^{-S_{\mathcal{G}}[g]} < \infty. \quad (3.8)$$

We implement the normalization

$$\det \tilde{G} = \det G, \quad \det \tilde{g} = 1 \quad (3.9)$$

and obtain

$$d\Psi = \left[-\tilde{G}^{-1} \frac{\delta S^{\text{tot}}}{\delta \Psi} + \frac{1}{\sqrt{\det \tilde{G}}} \frac{\delta(\tilde{G}^{-1} \sqrt{\det \tilde{G}})}{\delta \Psi} \right] ds + \tilde{E} dW \quad (3.10)$$

where S^{tot} denotes the total Yang-Mills action

$$S^{\text{tot}} = S + S_{\mathcal{G}}. \quad (3.11)$$

The equilibrium distribution of the Smoluchovski equation is obtained by direct inspection

$$\rho \longrightarrow \frac{\sqrt{\det \tilde{G}} e^{-S^{\text{tot}}}}{\int DB Dg \sqrt{\det \tilde{G}} e^{-S^{\text{tot}}}} = \frac{\sqrt{\det G} e^{-S^{\text{tot}}}}{\int DB Dg \sqrt{\det G} e^{-S^{\text{tot}}}}. \quad (3.12)$$

One recognizes equivalence to the Faddeev–Popov formula (1.3) as all unconventional *finite* contributions arising from S_G drop out.

4 Canonical Stochastic Quantization

We consider scalar field theory with a standard Lagrangian \mathcal{L} . The 4 + 1-dimensional Lagrangian $\tilde{\mathcal{L}}$, the canonically conjugated fields π and the Hamiltonian H are introduced by

$$\tilde{\mathcal{L}} = \frac{1}{2} \left(\frac{\partial \phi}{\partial s} \right)^2 - \mathcal{L}, \quad \pi = \frac{\partial \tilde{\mathcal{L}}}{\partial \frac{\partial \phi}{\partial s}}, \quad H = \int d^4x \mathcal{H}, \quad \mathcal{H} = \frac{1}{2} \pi^2 + \mathcal{L}. \quad (4.1)$$

Canonical stochastic quantization is defined in terms of the Langevin equations

$$d\phi = \frac{\delta H}{\delta \pi} ds, \quad d\pi = \left(-\frac{\delta H}{\delta \phi} - \frac{\delta H}{\delta \pi} \right) ds + dW. \quad (4.2)$$

In the equilibrium limit the Gibbs measure emerges [de Alfaro, Fubini & Furlan, 1983], [Ryang, Saito & Shigemoto, 1985], [Horowitz, 1985] and

$$\langle f \rangle = \frac{\int D\phi D\pi e^{-H} f(\phi)}{\int D\phi D\pi e^{-H}} = \frac{\int D\phi e^{-S} f(\phi)}{\int D\phi e^{-S}}. \quad (4.3)$$

Discussing scalar QED in this canonical scheme [Glück + Hüffel, 2007b] an ambiguity in the definition of the kinetic part of the Hamiltonian can be resolved by choosing \tilde{g} as the metric tensor

$$\mathcal{H}_{inv} = \frac{1}{2} \pi \tilde{g}^{-1} \pi + \mathcal{L}_{inv}. \quad (4.4)$$

\mathcal{L}_{inv} denotes the gauge invariant Lagrangian of scalar QED, $\Phi = (A, \phi, \bar{\phi})$ and $\pi = (\pi_A, \pi_\phi, \pi_{\bar{\phi}})$ are the collections of all the fields and of all the canonically conjugated fields, respectively. The canonical Langevin equations are given by

$$d\Phi = \frac{\delta H_{inv}}{\delta \pi} ds, \quad d\pi = \left(-\frac{\delta H_{inv}}{\delta \Phi} - \tilde{g} \frac{\delta H_{inv}}{\delta \pi} \right) ds + dW \quad (4.5)$$

which are transformed into adapted coordinates $(\Phi, \pi) \longrightarrow (\Psi, \Pi)$. Again, an equivalence class of canonical Langevin equations can be studied, out of which a specific representative is chosen. We then obtain

$$d\Psi = \frac{\delta H_{tot}}{\delta \Pi} ds, \quad d\Pi = \left(-\frac{\delta H_{tot}}{\delta \Psi} - \tilde{G} \frac{\delta H_{tot}}{\delta \Pi} \right) ds + dW, \quad (4.6)$$

where

$$H_{tot} = H_{inv} + S_G \quad (4.7)$$

In the equilibrium limit

$$\rho \longrightarrow \frac{e^{-H_{tot}}}{\int D\Psi D\Pi e^{-H_{tot}}} \quad (4.8)$$

and we can show straightforwardly equivalence to the standard path integral formulation of scalar QED.

5 Nonlinear Brownian Motion and Stochastic Quantization

Specific nonlinear modifications of stochastic processes were introduced by [Schweitzer, Ebeling & Tilch, 1998] and worked out with far reaching consequences [Schweitzer, 2003], [Ebeling & Sokolov, 2005]. Here we report on new applications to quantum field theory within the canonical stochastic quantization scheme.

The canonical Langevin equations are coupled to an “internal” energy E

$$d\phi = \frac{\delta H}{\delta \pi} ds, \quad d\pi = \left(-\frac{\delta H}{\delta \phi} - \frac{\delta H}{\delta \pi} + c E \frac{\partial V}{\partial \phi} \right) ds + dW, \quad (5.1)$$

which obeys

$$\frac{\partial E}{\partial s} = c_1 - c_2 E - c_3 E V. \quad (5.2)$$

We introduced various constants c, c_1, c_2, c_3 , as well as the potential $V(\phi)$

$$H = \int d^4x \left(\frac{1}{2} \pi^2 + \frac{1}{2} (\partial\phi)^2 + V(\phi) \right). \quad (5.3)$$

Similarly as [Schweitzer, Ebeling & Tilch, 1998] we assume that the evolution of the “internal” energy takes place at a much shorter time scale than that of the other fields. As a consequence we set $\frac{\partial E}{\partial s} = 0$ and express E in terms of ϕ . This gives rise to nonlinear modifications of the canonical Langevin equations. Performing a small coupling expansion we derive

$$\rho \longrightarrow \frac{e^{-\int d^4x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\partial\phi)^2 - aV(\phi) + bV^2(\phi) \right)}}{\int D\phi D\pi e^{-\int d^4x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\partial\phi)^2 - aV(\phi) + bV^2(\phi) \right)}}, \quad a, b > 0. \quad (5.4)$$

We remark that even for free scalar fields the nonlinear modifications induce interaction terms; a Mexican hat potential arises and the fields acquire nonzero vacuum expectation values.

Generalizing the nonlinear stochastic quantization procedure to scalar QED, we can construct the symmetry breaking potential of the Higgs mechanism [A. Glück and H. Hüffel, 2007a].

Finally, in a more refined analysis one could also try to solve numerically the Fokker Planck equation associated to the coupled system of partial differential equations given above without eliminating the “internal” energy E . This could lead to an elaborated picture of dynamical symmetry breaking and a deeper understanding of the quantum field theory vacuum structure.

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